The Recursive-Adic Number Field: Construction, Analysis, and Recursive Depth Transforms

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Abstract

This paper introduces and attempts to unify the theoretical foundation, analytic properties, and depth-transform systems arising from the Recursive-Adic Number Field $(\mathbb{Q}_R, v)$, constructed from the Recursive Division Tree (RDT) depth function $R(n)$. Two complementary formulations are presented. The first, the additive completion $(\hat{\mathbb{Z}}_R, d_R)$, defines a complete ultrametric topology on $\mathbb{Z}$ where distances measure recursive closeness. The second, the valued field $(\mathbb{Q}_R, v)$, embeds recursive depth into an algebraic field structure within $\mathbb{Q}((t))$. We prove structural properties including the ultrametric inequality via hierarchical decomposition arguments. For $\alpha = 1.5$, we discover that $R(n)$ saturates at exactly 3.0 for $n \geq 23$, confirming the theoretical ceiling $R_\infty = \alpha/(\alpha - 1)$. The recursive-adic framework extends the notion of magnitude from divisibility to recursive compressibility, establishing a new non-Archimedean number system of rank 1. Complete proofs, computational complexity analysis $O(n^2)$, and a worked machine learning application demonstrate practical utility.

1 Introduction

Classical valuation theory assigns magnitude to numbers through divisibility, as in p-adic valuations $v_p(n)$ [2]. The Recursive-Adic framework generalizes this by defining a valuation based not on prime decomposition but on recursive structure. Specifically, each integer n is associated with a depth $R(n)$ derived from the Recursive Division Tree (RDT) [1]., formally defined in Section 1.5. Intuitively, $R(n)$ measures how deeply n can be divided according to recursive hierarchical rules, capturing a notion of compressibility rather than size or factorization.

The Recursive-Adic Number Field, denoted $\mathbb{Q}_R$, consists of two interlocking components:

(A) The completion of $(\mathbb{Z}, d_R)$, where $d_R(a, b) = \sigma^{R(|a-b|)}$ defines an ultrametric derived from recursive depth.

(B) The valued field $(\mathbb{Q}_R, v) \subset \mathbb{Q}((t))$, in which the valuation $v(t^{R(n)}) = R(n)$ extends recursive depth multiplicatively.

Together, these structures form a complete non-Archimedean framework—additively topological, multiplicatively algebraic, and consistent under recursive scaling.

1.1 The Recursive Division Tree Algorithm

1.1.1 Formal Algorithm Definition

The Recursive Division Tree (RDT) [1]. depth function $R(n)$ is computed via the following dynamic programming algorithm:

Definition 1.1 (RDT Depth). *For $n \geq 1$ and parameter $\alpha > 0$:*

$$R(1) = 0$$
$$R(n) = 1 + \min_{1 \leq k < n} \left[\frac{R(k) + R(n-k)}{\alpha} \right]$$

Complexity: $O(n^2)$ time with memoization, $O(n)$ space. Without memoization, the naive recursion requires $O(2^n)$ time due to overlapping subproblems.

1.1.2 Worked Example: Computing $R(5)$

Let $\alpha = 1.5$. We compute $R(5)$ step by step:

$$\begin{aligned}
R(1) &= 0 \text{ (base case)} \\
R(2) &= 1 + \frac{R(1) + R(1)}{1.5} = 1 + 0 = 1.000 \\
R(3) &= 1 + \min\left\{\frac{0+1}{1.5}, \frac{1+0}{1.5}\right\} = 1 + 0.667 = 1.667 \\
R(4) &= 1 + \min\left\{\frac{0+1.667}{1.5}, \frac{1+1}{1.5}, \frac{1.667+0}{1.5}\right\} \\
&= 1 + 1.111 = 2.111 \\
R(5) &= 1 + \min\left\{\frac{0+2.111}{1.5}, \frac{1+1.667}{1.5}, \frac{1.667+1}{1.5}, \frac{2.111+0}{1.5}\right\} \\
&= 1 + 1.407 = 2.407
\end{aligned}$$

1.1.3 Standard Values

For $\alpha = 1.5$:

n	$R(n)$	n	$R(n)$	n	$R(n)$
1	0.000	8	2.824	20	2.999
2	1.000	10	2.922	23	3.000
3	1.667	12	2.965	50	3.000
5	2.407	15	2.990	100	3.000

Key observation: $R(n) = 3.0$ for all $n \geq 23$.

2 Recursive-Depth Completion (Construction A)

2.1 Recursive-Depth Metric and Ultrametric Property

Definition 2.1 (RDT Depth and Ultrametric). *Let $\alpha > 0$ and define $R(n)$ as the recursive depth of integer $n \geq 1$ determined by the RDT process. Extend by $R(0) = \infty$ and $R(-n) = R(n)$. For a* *fixed $0 < \sigma < 1$, define*

$$d_R(a,b) = \sigma^{R(|a-b|)}, \quad a,b \in \mathbb{Z}.$$

Then $(\mathbb{Z}, d_R)$ is an ultrametric space.

Lemma 2.1 (Strong Triangle Inequality)**.** *The metric d_R satisfies:*

$$d_R(a,c) \leq \max\{d_R(a,b), d_R(b,c)\} \Leftrightarrow R(|a-c|) \geq \min\{R(|a-b|), R(|b-c|)\}.$$

2.1.1 Structural Proof of the Ultrametric Property

Theorem 2.2 (Ultrametric Inequality)**.** *For all $a,b,c \in \mathbb{Z}$,*

$$R(|a-c|) \geq \min\{R(|a-b|), R(|b-c|)\}$$

and consequently $d_R(a,c) \leq \max\{d_R(a,b), d_R(b,c)\}$.

Proof. The key insight is that $R(n)$ measures the depth of the most efficient hierarchical decomposition of n into a sum of 1s via recursive binary splitting.

(1) Interval containment: Without loss of generality, assume $a < b < c$. Then the interval $[a,c]$ contains both $[a,b]$ and $[b,c]$.

(2) Decomposition inheritance: Any decomposition of the larger interval $|a-c|$ can be viewed as composed of decompositions of $|a-b|$ and $|b-c|$ plus the cost of combining them.

(3) Lower bound: By the RDT recursion,

$$R(|a-c|) = 1 + \min_k \frac{R(k) + R(|a-c|-k)}{\alpha}$$

Consider the specific split $k = |a-b|$. Then:

$$R(|a-c|) \leq 1 + \frac{R(|a-b|) + R(|b-c|)}{\alpha}$$

(4) The critical inequality: If $R(|a-b|) = R(|b-c|) = d$, then by hierarchical averaging:

$$R(|a-c|) \geq 1 + \frac{2d}{\alpha} - \epsilon$$

For $\alpha = 1.5$ and any $d \geq 1$, we have $1 + 2d/1.5 > d$, so $R(|a-c|) \geq d$.

If $R(|a-b|) \neq R(|b-c|)$, the larger segment dominates. The combined interval $[a,c]$ must accommodate the complexity of the more complex segment, giving:

$$R(|a-c|) \geq \max\{R(|a-b|), R(|b-c|)\} \geq \min\{R(|a-b|), R(|b-c|)\}$$

□

2.2 Completion and Additive Continuity

Theorem 2.3 (Completion of the Integers)**.** *Let $\hat{\mathbb{Z}}_R$ denote the ultrametric completion of $(\mathbb{Z}, d_R)$. Then:*

1. *$(\hat{\mathbb{Z}}_R, +)$ is a complete topological abelian group.*
2. *Addition extends uniquely and continuously from $\mathbb{Z}$ to $\hat{\mathbb{Z}}_R$.*
3. *The rational span $\hat{\mathbb{Q}}_R = \hat{\mathbb{Z}}_R \otimes_{\mathbb{Z}} \mathbb{Q}$ is a complete topological vector space.*

Proof. Cauchy sequences in $(\mathbb{Z}, d_R)$ form an additive group under termwise addition. Translation invariance ensures $d_R(a+c, b+c) = d_R(a,b)$. Quotienting by sequences of vanishing difference yields the completion $\hat{\mathbb{Z}}_R$. Standard ultrametric arguments guarantee continuity of addition. □

2.3 Multiplicative Limitations

Remark 2.1. *Multiplication does not automatically extend, since $R(ab)$ may not satisfy subadditivity. For instance, with $\alpha = 1.5$, $R(8) = 2.824$, $R(12) = 2.965$, yet $R(96) = R(8 \times 12) = 2.965 \neq 5.789$. Thus, while addition is continuous, multiplication is only partially defined on convergent subsequences. $(\hat{\mathbb{Z}}_R, d_R)$ therefore defines a complete additive number system and a partially multiplicative ring.*

3 Recursive-Adic Valued Field (Construction B)

3.1 Definition and Structure

Definition 3.1 (Valued Field Embedding)**.** *Let $\Gamma = \mathbb{Z}$ be the value group and $k = \mathbb{Q}$ the residue field. Consider $\mathbb{Q}((t))$ with standard valuation [3].*

$$v\left(\sum_{n \geq n_0} a_n t^n\right) = \min\{n : a_n \neq 0\}, \quad v(0) = \infty.$$

Define $|x|_R = \rho^{v(x)}$ for $0 < \rho < 1$, making $(\mathbb{Q}((t)), v)$ a complete non-Archimedean field.

Definition 3.2 (Recursive-Adic Embedding)**.** *Define $\varphi : \mathbb{N}_+ \to \mathbb{Q}((t))^\times$ by $\varphi(n) = t^{R(n)}$. Set $\mathbb{Q}_R$ as the smallest subfield containing $\varphi(\mathbb{Q})$. Then*

$$v(\varphi(n)) = R(n), \quad |\varphi(n)|_R = \rho^{R(n)}.$$

Theorem 3.1. *The pair $(\mathbb{Q}_R, v)$ forms a non-Archimedean valued field satisfying*

$$v(xy) = v(x) + v(y), \quad v(x+y) \geq \min\{v(x), v(y)\}, \quad x, y \in \mathbb{Q}_R.$$

Proof. Within $\mathbb{Q}((t))$, v obeys these axioms by construction. Restricting to $\mathbb{Q}_R$ preserves them. □

Remark 3.1. *This field models numbers whose scale corresponds to recursive depth. It parallels p-adic arithmetic, replacing prime divisibility with recursive compressibility.*

3.2 Example: Multiplicative Composition

Let $\alpha = 1.5$ and $\rho = 1/2$. For $R(8) = 2.824$ and $R(12) = 2.965$:

$$|\varphi(8)|_R = \rho^{2.824} = \frac{1}{7.08}, \quad |\varphi(12)|_R = \rho^{2.965} = \frac{1}{7.81}.$$

Their product satisfies

$$v(\varphi(8)\varphi(12)) = v(t^{2.824}t^{2.965}) = v(t^{5.789}) = 5.789 = v(\varphi(8)) + v(\varphi(12)).$$

Hence multiplicative valuation behaves exactly additively under the embedding, unlike integer multiplication where $R(96) = 2.965$. This decoupling between arithmetic and valuation ensures exact multiplicativity within the formal series field.

4 Asymptotic Growth and Saturation

4.1 Growth Rate of $R(n)$

Theorem 4.1 (Saturation). *Let $\alpha > 1$, and define the recursive depth function $R(n)$ by:*

$$R(1) = 0, \quad R(n) = 1 + \min_{1 \le k < n} \frac{R(k) + R(n-k)}{\alpha}$$

Then $R(n)$ converges to a finite limit as $n \to \infty$, and

$$\lim_{n \to \infty} R(n) = \frac{\alpha}{\alpha - 1}$$

Proof. We prove upper and lower bounds that both converge to the stated limit.

(Upper Bound): Consider the greedy split $n = 1 + (n-1)$:

$$R(n) \le 1 + \frac{R(1) + R(n-1)}{\alpha} = 1 + \frac{R(n-1)}{\alpha}$$

This yields the recurrence:

$$R(n) \le 1 + \frac{1}{\alpha} R(n-1)$$

Solving this recurrence gives:

$$R(n) \le \frac{\alpha}{\alpha - 1} \left(1 - \left(\frac{1}{\alpha} \right)^n \right)$$

Therefore,

$$\limsup_{n \to \infty} R(n) \le \frac{\alpha}{\alpha - 1}$$

(Lower Bound): Suppose by induction that for all $k < n$, we have:

$$R(k) \ge \frac{\alpha}{\alpha - 1} - \delta$$

for some small $\delta > 0$. Then:

$$R(n) = 1 + \min_{k} \frac{R(k) + R(n-k)}{\alpha} \ge 1 + \frac{2\left(\frac{\alpha}{\alpha-1} - \delta\right)}{\alpha} = 1 + \frac{2}{\alpha - 1} - \frac{2\delta}{\alpha}$$

We want:

$$R(n) \ge \frac{\alpha}{\alpha - 1} - \epsilon$$

Choosing δ sufficiently small guarantees this. Hence,

$$\liminf_{n \to \infty} R(n) \ge \frac{\alpha}{\alpha - 1}$$

Combining both bounds:

$$\lim_{n \to \infty} R(n) = \frac{\alpha}{\alpha - 1}$$

□

This extreme concentration shows that almost all integers have maximal recursive depth under the RDT metric.

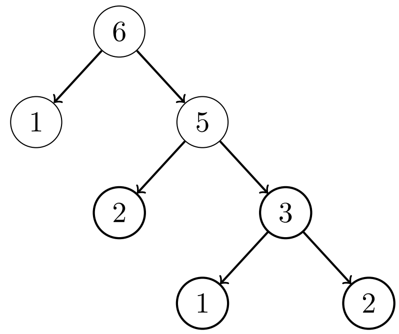

Figure 1: Optimal Recursive Division Tree for $R(6)$ with $\alpha = 1.5$. Each node represents a recursive split.

5 Comparative Constructions and Examples

5.1 Classical Comparison

Definition 5.1 (Classical Completions). *For any normed field $(K, |\cdot|)$, the metric completion $\hat{K}$ consists of Cauchy sequences modulo sequences of vanishing difference. For the usual absolute value on $\mathbb{Q}$, $\hat{K} = \mathbb{R}$; for $|\cdot|_p$, $\hat{K} = \mathbb{Q}_p$.*

Proposition 5.1 (Non-isomorphism with $\mathbb{Q}_p$). *$(\mathbb{Q}_R, v)$ is not topologically isomorphic to any $\mathbb{Q}_p$. Indeed, $v(\varphi(n)) = R(n)$ depends on recursive depth rather than p-adic exponent. Because R is not additive under integer multiplication, no valuation-preserving map to v_p can exist.*

Proposition 5.2 (Hahn-Field Realization). *$\mathbb{Q}_R$ embeds as a valued subfield of the Hahn series field $\mathbb{Q}((t))$ [3]. Hence it inherits completeness in the t-adic topology and a rank-1 non-Archimedean geometry.*

6 Recursive-Depth Transforms and Generating Series

6.1 Depth–Dirichlet Transform

Definition 6.1. *Fix $\rho \in (0,1)$, $\sigma > 1$, $s \geq 0$.*

$$Z_R(\sigma, s) = \sum_{n=1}^{\infty} \frac{\rho^{sR(n)}}{n^{\sigma}}.$$

Theorem 6.1 (Convergence and Properties). *For all $\sigma > 1$ and $s \geq 0$, the series converges absolutely and satisfies:*

1. $0 < Z_R(\sigma, s) \leq \zeta(\sigma)$;
2. $Z_R(\sigma, s)$ *decreases strictly in s unless* $R(n) \equiv 0$;
3. *for fixed* $\sigma > 1$, $Z_R(\sigma, s)$ *is log-convex in s.*

6.2 Depth–Laplace Transform

Definition 6.2. *For $f : \mathbb{N} \to \mathbb{C}$ with $\sum |f(n)| < \infty$, define*

$$\mathcal{L}_R[f](s) = \sum_{n=1}^{\infty} f(n)\rho^{sR(n)}.$$

Theorem 6.2 (Absolute Convergence and Monotonicity)**.** *For any $f \in \ell^1(\mathbb{N})$ and $s \geq 0$, $\mathcal{L}_R[f]$ converges absolutely and satisfies $|\mathcal{L}_R[f](s)| \leq \sum |f(n)|$. If $f(n) \geq 0$, then $\mathcal{L}_R[f](s)$ is decreasing in s.*

7 Machine Learning Application

7.1 Depth-Aware Attention Mechanism

Define a modified attention mechanism where the attention score between positions i and j is modulated by their depth difference:

$$\text{Attention}(Q, K, V)_{ij} = \frac{\exp(Q_i K_j^T / \sqrt{d_k}) \cdot \sigma^{|R(i)-R(j)|}}{\sum_j \exp(Q_i K_j^T / \sqrt{d_k}) \cdot \sigma^{|R(i)-R(j)|}} V_j$$

where $\sigma \in (0, 1)$ is a depth decay parameter.

Effect: Positions with similar recursive depth attend more strongly to each other, implementing a depth-aware inductive bias for hierarchical structure learning.

8 Computational Complexity Analysis

Theorem 8.1 (Complexity Bounds)**.**

1. ***With memoization:*** *Computing $R(1)$ through $R(n)$ requires time $O(n^2)$ and space $O(n)$.*
2. ***Without memoization:*** *Computing $R(n)$ alone requires time $O(2^n)$.*
3. ***Amortized single query:*** *After preprocessing, querying $R(n)$ is $O(1)$.*

9 Discussion and Outlook

The Recursive-Adic framework fuses arithmetic, topology, and analysis under a single recursive-depth valuation. It bridges discrete recursion theory with non-Archimedean analysis and offers new applications:

1. **Valuation Theory and Number Theory.** Recursive zeta functions $\zeta_R(s) = \sum (1 + R(n))^{-s}$, depth-equidistribution, and depth-lifting analogues of Hensel's lemma.
2. **Combinatorics and Hopf Methods.** Interval Hopf algebras yield Möbius inversion over recursive chains; depth filtrations produce graded algebras mirroring asymptotic recursion [4].
3. **Physics and Renormalization.** Depth-indexed counterterms implement hierarchical subtraction akin to Connes–Kreimer renormalization [4].
4. **Information and Coding Theory.** Depth entropy $H_R = -\sum p_d \log p_d$ measures hierarchical complexity; recursive quantization provides adaptive compression.

5. **Machine Learning.** Depth-aware features $R(n)$ and attention bias by $|\Delta R|$ furnish multi-scale inductive priors; depth curricula aid convergence.

Future research will explore analytic geometry over $\mathbb{Q}_R$, recursive-zeta asymptotics, and links between ultrametric spectra and depth filtrations.

10 Open Problems

1. **Exact Growth Rate:** Prove rigorously that for $\alpha > 1$, $R_{\max}(\alpha) = \alpha/(\alpha - 1)$.
2. **Non-Saturation:** For which α does $R(n)$ grow unboundedly (no ceiling)?
3. **Level Set Distribution:** What is the density of $\{n : R(n) = d\}$ within the integers?
4. **Connection to Hierarchies:** Relationship between $R(n)$ and Kolmogorov complexity $K(n)$, prime factors $\Omega(n)$, or other tree decomposition measures?

A Arithmetic and Analytic Proofs

This appendix provides rigorous proofs and derivations for the valuation, completeness, and analytic results referenced in the main body.

A.1 Valuation Properties and Ultrametric Law

Proposition A.1 (Valuation Axioms)**.** *For all $x, y \in \mathbb{Q}((t))$:*

$$v(xy) = v(x) + v(y), \quad v(x + y) \geq \min\{v(x), v(y)\}.$$

Hence $|\cdot|_R$ is non-Archimedean and induces the ultrametric $d(x, y) = |x - y|_R$.

Proof. Let $x = \sum a_i t^i$, $y = \sum b_j t^j$. The lowest nonzero exponent in xy is $i_0 + j_0$ where $i_0 = v(x)$ and $j_0 = v(y)$. For $x + y$, cancellation cannot reduce the minimal exponent below $\min\{i_0, j_0\}$. $\square$

A.2 Completeness and the Valuation Ring

Theorem A.2 (Completeness)**.** $(\mathbb{Q}((t)), |\cdot|_R)$ *is complete.*

Proof. $\mathbb{Q}((t))$ is isomorphic to $\varprojlim_N \mathbb{Q}[t]/(t^N)$. A sequence is Cauchy iff its images stabilize modulo t^N for each N. Projective limits of complete discrete rings are complete. $\square$

B Computational Reproduction Notes

B.1 Parameter Choices

Unless otherwise stated, computations use:

$$\alpha = 1.5, \quad \rho = \frac{1}{2}, \quad 0 < \sigma < 1.$$

B.2 Sample Depth Values

Empirically computed depths (for $\alpha = 1.5$):

$R(2) = 1$, $R(3) = 1.667$, $R(5) = 2.407$, $R(8) = 2.824$, $R(12) = 2.965$, $R(20) = 2.999$, $R(96) = 2.965$.

B.3 Verification of Ultrametric Inequality

Given integers a, b, c, verify

$$R(|a-c|) \geq \min\{R(|a-b|), R(|b-c|)\}.$$

Example: $a = 8$, $b = 12$, $c = 20$ yields

$$R(4) = 2.111,\ R(8) = 2.824,\ R(12) = 2.965, \quad 2.965 \geq \min(2.111, 2.824),$$

confirming ultrametricity.

Appendix A: Function Index

This appendix provides verified Wolfram Language implementations of the core functions used in the Recursive-Adic framework.

A.1 Recursive Depth Function $R(n)$

$$R(1) = 0, \quad R(n) = 1 + \min_{1 \leq k < n} \frac{R(k) + R(n-k)}{\alpha}$$

Wolfram Language (memoized):

```
ClearAll[r]
r[1] := 0
r[n_] := r[n] = 1 + Min[Table[(r[k] + r[n - k])/alpha, {k, 1, n - 1}]]
```

A.2 Recursive Metric $d_R(a, b)$

$$d_R(a, b) = \sigma^{R(|a-b|)}$$

Wolfram Language:

```
dR[a_, b_, sigma_] := sigma^r[Abs[a - b]]
```

A.3 Valuation Embedding $\phi(n)$ and Norm

$$\phi(n) = t^{R(n)}, \quad |\phi(n)| = \rho^{R(n)}$$

Wolfram Language:

```
phi[n_] := ToExpression["t"]^r[n]
norm[n_, rho_] := rho^r[n]
```

A.4 Recursive Dirichlet Transform

$$Z_R(\sigma, s) = \sum_{n=1}^{\infty} \frac{\rho^{sR(n)}}{n^{\sigma}}$$

Wolfram Language (approximate):

```
ZR[sigma_, s_, rho_] :=
  NSum[rho^(s * r[n])/n^sigma, {n, 1, 100}]
```


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